Dynamic analysis of the effect of immigration on the demographic background of the pay-as-you-go pension system

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Problem Setting and motivation

Demographic equilibrium as a key variable to ensure the sustainability of a pay-as-you-go pension system.

“Ideal” shape
Problem Setting and motivation

The problem is that population pyramids of mature economies often display “critical” features

“Bad” shape

...for too many pensioners

“Few” contributors...
Problem Setting and motivation

With respect to previous estimates, ISTAT forecasts that the Italian population will grow more than 6 million people as a consequence of immigration.

In 2051 immigrants will represent 16.1%-18.4% of the whole population.

The basic idea:
Immigrants as a resource for stabilizing the population distribution in order to achieve the sustainability of the pay-as-you-go pension system.
Problem Setting and motivation

Pros

Rejuvenating the age structure of the population owing to two main reasons:

- Immigrants are generally young (*immediate effect*)
- Immigrants generally display fertility rates higher than the Italian one (*postponed effect*)

<table>
<thead>
<tr>
<th>ISTAT Fertility rates (mid scenario)</th>
<th>2007</th>
<th>2051</th>
</tr>
</thead>
<tbody>
<tr>
<td>Italian</td>
<td>1.36</td>
<td>1.39</td>
</tr>
<tr>
<td>Immigrants</td>
<td>2.35</td>
<td>1.86</td>
</tr>
</tbody>
</table>

*Remark*

The analysis of the demography for the pension system is the very first step towards a more general solution also including the economic component.
The basic population model

Consider a population without sex structure where:

- $\mathcal{N} \in \mathbb{N}$ is the upper bound of the age of an individual;
- $x_i(t)$ is the number of individuals of age belonging to $[i, i+1]$ (for $i = 0, 1, \ldots, \mathcal{N} - 1$) at time $t$;
- $\alpha_i \geq 0$ average per capita birth rate in the $i$-th age group;
- $0 < \omega_i < 1$ is the survival rate from age group $i$ to $i+1$;

The population vector results

$$x(t) = [x_0(t), x_1(t), \ldots, x_{N-1}(t)]^T$$

and the system matrix is

$$L = \begin{bmatrix}
\alpha_0 & \alpha_1 & \ldots & \alpha_{N-2} & \alpha_{N-1} \\
\omega_0 & 0 & \ldots & 0 & 0 \\
0 & \omega_1 & 0 & \ldots & 0 \\
0 & 0 & \omega_{N-2} & 0 & \ldots
\end{bmatrix}$$

The population dynamics (so called Leslie model) is

$$x(t + 1) = L \ x(t)$$
The modified population model

The Leslie model \( \rightarrow \) Insects

Our modified Leslie model \( \rightarrow \) Human population

Let

- \( N \) be the maximal age for each sex;
- \( x^F(t) \) be the female population vector;
- \( x^M(t) \) be the male population vector;
- \( x(t) \) be the whole population vector;
- \( f_i \) be the per mil fertility;
- \( q_i^F \) be the female mortality rate;
- \( \omega_i^F = 1 - q_i^F \) be the female survival rate;
- \( \varphi = \frac{x_0^F(T)}{x_0^F(T) + x_0^M(T)} = \frac{x_0^F(T)}{x_0(T)} \) be the sex ratio (at birth)

The female per capita birth rate is \( \alpha_i^F = \frac{f_i}{1000} \varphi \) (i=15,...,50)
The modified population model

The modified Leslie female matrix becomes

\[
L^F = \begin{bmatrix}
0 & 0 & \ldots & \alpha_{15}^F & \ldots & \alpha_{50}^F & 0 & \ldots & 0 & 0 \\
\omega_0^F & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & \omega_1^F & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix}
\]

The dynamics of females reads as \[x^F(t+1) = L^F x^F(t)\]

The modified Leslie male matrix becomes

\[
L^M = \begin{bmatrix}
0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\omega_0^M & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & \omega_1^M & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix}
\]

The dynamics of males reads as \[x^M(t+1) = L^M x^M(t) + \frac{(1-\varphi)}{\varphi} e_1 \circ L^F x^F(t)\]
A stabilization theorem

Setting $d = \begin{bmatrix} 0 & \ldots & 0 & \frac{1-\phi}{\phi} \alpha_1^F & \ldots & \frac{1-\phi}{\phi} \alpha_N^F & 0 & \ldots & 0 \end{bmatrix}$ and denoted by $\Lambda = \begin{bmatrix} L^F & O \\ d & 0 \\ O & L_1^M \end{bmatrix}$

the block matrix synthesizing the whole population ($L_1^M$ is the $(N-1) \times N$ matrix obtained by cutting the first row of $L^M$), the dynamics is

$$x(t+1) = \Lambda x(t)$$

Problem

The matrix $\Lambda$ is neither irreducible nor primitive and hence the Perron-Frobenius conditions ensuring the existence of a dominant positive eigenvalue are not satisfied.

Solution

We prove that sub-matrix $L_{50}^F = \begin{bmatrix} 0 & 0 & \ldots & \alpha_1^F & \ldots & \alpha_{49}^F & \alpha_{50}^F \\ \omega_0^F & 0 & \ldots & 0 & \ldots & 0 & 0 \\ 0 & \omega_1^F & \ldots & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & \omega_{49}^F & 0 \end{bmatrix}$ is irreducible

and primitive. Applying Perron-Frobenius to $L_{50}^F$ we get the asymptotical distribution of the sub-population and prove that it can be extended to the whole population.
A stabilization theorem

We state the following

**Theorem A (2006)**
There exists a population state $x_0$, which is a nonnegative eigenvector of the matrix $\Lambda$, associated with a positive eigenvalue $\lambda_0$. On the long term, the population age distribution tends to the equilibrium age distribution in the sense that, for any initial state $x(0)$, there exists $s > 0$ such that

$$\lim_{t \to \infty} \frac{x(t)}{\lambda_0^t} = sx^0$$

Angrisani, Attias, Bianchi, Varga, “Immigration and demographic equilibrium”
Adding immigration

**Purposes**

- Equipping the model with immigration
- Setting up a dynamic model in order to steer the population by optimizing the immigration flow with respect to the pension system requirements

Denoted by:

- \( \alpha'_i \) the immigrant female reproduction rate
- \( y_i^F(t) [y_i^M(t)] \) the immigrant female [male] population vector

one has

**Basic assumption:** \( \alpha'_i \geq \bar{\alpha}_i(0) := \alpha_i^F \quad (i = 15, \ldots, 50) \)

- the female reproduction rate dynamics

\[
\bar{\alpha}_i(t + 1) := \frac{\alpha_i^F x_i^F(t)}{\omega_i^F x_i^F(t) + y_i^F(t)} \bar{\alpha}_i(t) + \frac{y_i^F(t)}{\omega_i^F x_i^F(t) + y_i^F(t)} \alpha'_i
\]
Adding immigration

- the matrix 
  \[ \Lambda(t) := \begin{bmatrix} \bar{L}^F(t) & 0 \\ \bar{d}(t) & 0 \\ 0 & L^M_t \end{bmatrix} \]
  where the blocks depend on the new reproduction rates;

- the dynamics of population
  \[
  x^F(t+1) = \bar{L}^F(t)x^F(t) + y^F(t) \\
  x^M(t+1) = \bar{L}^F(t)x^M(t) + \frac{(1-\phi)}{\phi}L^F x^F(t) + y^M(t)
  \]
  or, in terms of the whole population, setting \( y = [y^F \ y^M]^T \)
  \[
  x(t+1) = \Lambda(t)x(t) + y(t)
  \]
Controlling the population to demographic equilibrium

Purpose
Finding an immigration vector $y(0)$ s.t. the new population state
$$x(1) := \Lambda(0)x(0) + y(0)$$
provides an equilibrium age distribution for system matrix $\Lambda(0) = \Lambda$.

Since, by Theorem A, there exists a normalized positive eigenvector $z^{(0)}$, corresponding to the positive eigenvalue $\lambda_0$ of $\Lambda(0)$, we have to minimize in $u > 0$

$$\sum_k y_k(0) = \sum_k (uz^{(0)} - \Lambda(0)x(0))_k$$

s.t. $uz^{(0)} - \Lambda(0)x(0) \geq 0$

to it is easy to see that the constrained problem (1) is equivalent to the free problem

$$f_0(u) := \sum_k (uz^{(0)} - \Lambda(0)x(0))_k \to \min, \quad u \in \left[u_0 = \max_k \frac{(\Lambda(0)x(0))_k}{z^{(0)}_k}, +\infty\right]$$

Trivially, $u_0$ is the unique solution of problem (2).
Controlling the population to demographic equilibrium

Introducing a convergence parameter $\mu \in [0,1]$, and recursively solving the optimization problem (2) for increasing $t$, one gets the following

**Theorem B (2010)**

Sequences $(\overline{\Lambda}(t))$ and $(\lambda_i)$ are monotonically convergent, sequences $(z^{(i)})$ is convergent and for the respective limits $\Lambda^\infty = \lim(\overline{\Lambda}(0))$, $\lambda^\infty = \lim(\lambda_i)$ and $z^\infty = \lim(z^{(i)})$, we have $\Lambda^\infty z^\infty = \lambda^\infty z^\infty$.

**Remark 1.** For every $t$, the normed eigenvector of $\overline{L}_{50}(t)$ is unique, and its extension to $z^{(i)}$ is also unique. Therefore, the asymptotic equilibrium attained by immigration control is also uniquely determined.
Controlling the population to demographic equilibrium

**Theorem C** [Angrisani, Attias, Bianchi, Varga (2010)]
The age distribution of population converges to the equilibrium age distribution, i.e.

$$\lim_{t \to \infty} \left( \frac{x(t+1)}{u_t} \right) = z^\infty$$

**Remark 2.** Theorems B and C hold if \((\mu^i)\) is replaced by any positive sequence tending to zero. This choice makes the application of the model more flexible.
Immigration scenarios and simulations

• The control model is equipped with a parameter of convergence (between 0 and 1) that regulates the speed of convergence and the total immigration at the same time. A value near 1 slows down the convergence but limits the yearly admission of immigrants.

• In our simulations this parameter is set to 0.9; and the algorithm was modified, admitting only immigrants under age 35. This modification doesn’t change the convergence of the algorithm.
Immigration scenarios and simulations

- In the simulations we considered 3 immigration scenarios
  
  A) Strongly limited constant yearly immigration (180,000)
  
  B) Constant yearly immigration, corresponding to year 2008 (i.e. 494,289 new immigrants)
  
  C) Immigration controlled by our model, based on the algorithm moving the age distribution of the population towards a demographic equilibrium (eigenvector of the updated Leslie matrix)
Immigration scenarios and simulations

• In the model analysis we concentrate on the change of the inverse dependency ratio.

• In the following figures, for scenarios A), B) and C), we plot against time (years) the

  \[
  \text{Inverse old-age dependency ratio} = \frac{\text{Number of people aged 15-64}}{\text{Number of people aged 65 and over}}
  \]

  and the

  \text{Total population size,}

  then, for scenario C), the

  \text{Yearly total immigration.}
Scenario (A): constant yearly immigration - 180,000

Figure 1. Inverse old-age dependency ratio, plotted against time (in years)

Figure 2. Total population size, plotted against time (in years)
Scenario (B): constant yearly immigration - 494.289

Figure 3. Inverse old-age dependency ratio, plotted against time (in years)

Figure 4. Total population size, plotted against time (in years)
Scenario (C): Control model, initial immigration = 450.461

**Figure 5. Inverse old-age dependency ratio**, plotted against time (in years)

**Figure 6. Total population size**, plotted against time (in years)
Figure 7. Total population size, for scenario C), plotted against time (years) control model, initial immigration= 450.461, time horizon= 200 years
Figure 8. Yearly total immigration, for scenario C), showing long-term stabilization of immigration policy, initial immigration = 450.461,
Discussion

- Classical Leslie population growth model can be extended to a control-theoretical model, appropriate for dynamic simulation of the demographic background of a pension system.

- In the control model, the yearly immigration can be determined for each age class, such that
  - the age distribution of the population moves towards a demographic equilibrium,
  - each year the total immigration is minimized,
  - a parameter can be set to regulate the speed of convergence and the total immigration at the same time.

- The efficiency of the algorithm was shown, by comparing it to constant immigration scenarios, in terms of the active/pensioner ratio.
References


• Attias A., Equilibrio demografico e flessibilità del sistema pensionistico, Atti del XIII Convegno di Teoria del Rischio (2007), pp. 5-37


• Leslie, P. H. (1948). Some further notes on the use of matrices in certain population mathematics. Biometrika, 35, No 3-4, 213-245